INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, First Semester, 2005-06
Statistics - I, Semesteral Examination, 22 November 2005
Calculator and Statistical Tables may be used
Marks are shown to the left of each question

(5) 1. Suppose $T \sim \text{Exp}(\lambda)$ with density

$$f_T(t) = \begin{cases} \lambda \exp(-\lambda t), & \text{if } t > 0; \\ 0 & \text{otherwise.} \end{cases}$$

Define a random variable X by

$$X = k \quad \text{if} \quad k-1 < T \leq k \quad \text{for} \quad k = 1, 2, \ldots$$

What is a method of moments estimator of λ based on a random sample $X_1, \ldots, X_n$?

(5) 2. Let $Z_1, Z_2, \ldots, Z_p$ be i.i.d $N(0,1)$ for $p > 2$ and $0 < \rho < 1$ be a constant. Define

$$X_1 = Z_1, \quad X_2 = \sqrt{1-\rho^2} Z_2 + \rho X_1, X_{i+1} = \sqrt{1-\rho^2} Z_{i+1} + \rho X_i, \quad i = 2, \ldots, p-1.$$

Find the joint distribution of $X_1, \ldots, X_p$.

(8) 3. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. Define

$$Y_1 = \frac{1}{6}\sum_{i=1}^{6} X_i, Y_2 = \frac{1}{7}\sum_{i=7}^{13} X_i, S_1 = \sum_{i=1}^{6}(X_i - Y_1)^2, S_2 = \sum_{i=7}^{13}(X_i - Y_2)^2, \text{ and } T = S_1/S_2.$$

(a) Find $P(5/6 < T \leq 5)$.
(b) Find $P(5S_2 \leq 6S_1)$.

(8) 4. Let $X_1, \ldots, X_9$ be a random sample from $N(\mu, 1)$. It is of interest to test $H_0 : \mu = 0$ against $H_1 : \mu > 0$. Consider the test which rejects H_0 if $3\overline{X} > 2.326$.
(a) What is α, the level of significance of this test? What is the $P($ Type I Error $)$ for this test?
(b) What is the power of the test at $\mu = 1.425$?

(6) 5. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$, where σ^2 is known. Let $0 < \alpha < 1$.
(a) What is a 100(1-α)% confidence interval for μ?
(b) Show that testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ at the α level of significance is the same as constructing a 100(1-α)% confidence interval for μ and then checking whether μ_0 is inside this interval.

(10) 6. X is said to have the lognormal distribution with parameters μ and σ^2 if $Y = \log(X) \sim N(\mu, \sigma^2)$. (Here log is the natural logarithm.) The notation $X \sim \text{Lognormal}(\mu, \sigma^2)$ is used here.
(a) Find the density f_X.
(b) How does one use the rejection method to generate a Lognormal(μ, σ^2) random variate using U(0, 1) random variates, when μ and σ^2 are specified?

(8) 7. Consider the linear model $y_i = \beta x_i + \epsilon_i$, $i = 1, \ldots, n$ for the data $(x_1, y_1), \ldots, (x_n, y_n)$. Suppose the ϵ_i have mean 0, variance $\sigma^2 > 0$, and they are uncorrelated.
(a) Find the least squares estimate for β.
(b) Find the expectation and mean square error of the estimate in (a) above assuming that the linear model is true.